INTEGRALS IN AN INFINITE NUMBER OF DIMENSIONS.

By P. J. Daniell.

1. In a recent issue of the Annals of Mathematics* there appeared a paper by the author on "A General Form of Integral." In it a method was given whereby integrals could be defined for functions of general elements (p) which could theoretically be of any character. The author was then unable to give a definite example in which the elements (p) were points in a denumerably infinite number of dimensions. According to Hildebrandt,† a definite example, which does not reduce to an infinite series or to an integral over a finite number of dimensions, is still lacking. Even Fréchet‡ did not give any example which was sufficiently general. Since the publication of his previous paper, the author has found two examples, and this is now an account of them. The first is a generalization of the Lebesgue integral in the interval $(0 \leq x \leq 1)$; the second an infinitely multiple Stieltjes integral of positive type. The author hopes to publish soon a still wider generalization of the Stieltjes integral.

For the purpose before us it is necessary to use certain properties of points and functions in a denumerable number of dimensions. Some of these properties are obtained by Fréchet§ in his thesis to which reference will be made by means of notation F., p. 40, for example, referring to page 40.

The elements (p) are points in a denumerable number of dimensions, that is to say, having a denumerable number of coördinates,

$$p = (x_1, x_2, \cdots x_n, \cdots).$$

Fréchet (F., p. 40) defines the "écart" of two points p, p', to be

$$(p, p') = \frac{|x_1 - x_1'|}{1 + |x_1 - x_1'|} + \cdots + \frac{1}{n!}\frac{|x_n - x_n'|}{1 + |x_n - x_n'|} + \cdots,$$

and the class of points, for which this écart is defined, he calls (E_ω).

By F., p. 45, a function $f(p)$ of elements p of class (E_ω) is said to be *continuous*, if $\lim f(p_r) = f(p)$, as the sequence $\{p_r\}$ approaches p.

Daniell's theory was based on a class of functions, T_0, satisfying the following requirements: Each function must be bounded, and the class

* P. J. Daniell, Annals of Mathematics, vol. 19 (1918), p. 279.

† T. H. Hildebrandt, Bulletin of the American Mathematical Society, vol. 24 (1917), p. 116.

‡ M. Fréchet, Bulletin de la Société de France, vol. 43 (1915), p. 249.

§ M. Fréchet, Rendiconti di Circolo matematico di Palermo, vol. 22 (1906), pp. 1–74.

T_0 is closed with respect to the operations (op. C) multiplication by a constant, (op. A) addition of two functions, and "logical addition and multiplication" of two functions. It simplifies the theory, if we notice that when T_0 is closed with respect to (op. C) (op. A), then closure with respect to "logical addition and multiplication" is equivalent logically to closure with respect to (op. M), the operation of taking the modulus. For, on the one hand,

$$|f| = f \vee 0 - f \wedge 0,$$

and $0 = 0 \times f$ belongs to T_0; on the other hand,

$$f \vee g = \tfrac{1}{2}[f + g + |f - g|],$$

$$f \wedge g = \tfrac{1}{2}[f + g - |f - g|].$$

We shall choose the class T_0 to be the class of functions

$$f(p) = f(x_i, x_j, \cdots x_r),$$

which are functions of the finite number of variables, $x_i, x_j, \cdots x_r$, (chosen from the variables, $x_1, x_2, \cdots x_n, \cdots$) and bounded and continuous in the domain considered, that is to say, in the first case, in the finite domain $(0 \leq x_i \leq 1, \cdots 0 \leq x_r \leq 1)$, and in the second case, for all finite values of $x_i, x_j, \cdots, x_r$. If f and g are two such functions, e. g., $f(x_i, \cdots, x_r)$, $g(x_p, \cdots, x_t)$, their sum will be a function of $x_i, x_j, \cdots x_r, x_p, \cdots x_t$ (where some of these variables may be identical) and $f + g$ will be also bounded and continuous. The class T_0 will evidently satisfy all the required closure conditions.

2. **Simple Integral.** For any function

$$f(p) = f(x_i, x_j, \cdots, x_r)$$

of class T_0 we define

$$I(f) = \int_0^1 \cdots \int_0^1 f(x_i, \cdots x_r)dx_i dx_j \cdots dx_r.$$

This definition is possible since f is a continuous function of a finite number of variables. Then $I(f)$ is finite and satisfies the conditions,

$$(C) \qquad I(cf) = cI(f), \text{ if } c \text{ is any constant,}$$

$$(A) \qquad I(f_1 + f_2) = I(f_1) + I(f_2),$$

$$(P) \qquad I(f) \geq 0, \text{ if } f(p) \geq 0 \text{ for all } p.$$

We need to give an extended proof for (L) only, namely,

(L) If $f_1 \geq f_2 \geq \cdots \geq 0 = \lim f_n$ for every p,
then

$$\lim I(f_n) = 0.$$

In the first place,

$$|I(f)| \leq \max_{p} |f(p)|.$$

In this case $I(f_n) \leq \max f_n(p)$, or

$$\lim_{n=\infty} I(f_n) \leq \lim_{n=\infty} \max_{p} f_n(p).$$

The condition (L) will be satisfied if $\max f_n(p)$ approaches 0 with $1/n$. Assume that it does not, then we can find a $k > 0$, such that $\max f_n(p) \geq k$ for all n. But $f_n(p)$ is continuous and therefore attains its maximum at least once (F., p. 45), or the set E_n of points such that $f_n(p) \geq k$ contains at least one point. Moreover, since $f_{n+1}(p) \leq f_n(p)$ for all p, the set E_{n+1} is contained in the set E_n. The sets E_n are closed, for if $\{p_r\}$ $(r = 1, 2, \cdots)$ is a sequence of points contained in E_n, approaching p as a limit, then

$$f_n(p_r) \geq k, \qquad r = 1, 2, \cdots,$$

$$f_n(p) = \lim f_n(p_r) \geq k,$$

or p belongs to E_n.

The set $(0 \leq x_n \leq 1)(n = 1, 2, \cdots)$ is compact by F., p. 42, and it follows by F., p. 7, that there is at least one point common to every E_n, that is to say, there is a point p, such that $f_n(p) \geq k$ for all n. This is contrary to the hypothesis, that $\lim f_n(p) = 0$ for all p. Then our assumption must be incorrect, or

$$\lim_{n} \max_{p} f_n(p) = 0,$$

and thus (L) is proved. Our integral satisfies all the required conditions for functions of class T_0. We can then extend it to all summable functions according to the methods laid down by Daniell (l. c.). In particular, any function which can be obtained from functions of class T_0 by successive operations such as multiplication by constants, addition, forming the modulus and taking the limit of a sequence of functions bounded in their set, will be summable. For if K is any constant, $\varphi(p) = K$ is a member of class T_0, and summable; and we make use of Daniell, theorems (7·2, 7·3, 7·4, 7·7).

For example *any continuous function is summable.* For (1) it is bounded (F., p. 45) or a finite K can be found such that $|f(p)| \leq K$ for all p; (2) it is the limit of a sequence of functions of class T_0, bounded in their set.

If $p = (x_1, x_2, \cdots x_n, x_{n+1}, \cdots)$, consider the function

$$f_n(p) = f(x_1, x_2, \cdots x_n, 0, 0, \cdots).$$

This function is of class T_0 and $|f_n(p)| \leq K$, for all n and p.

If

$$p_n = (x_1, x_2, \cdots x_n, 0, 0, \cdots),$$

$$f_n(p) = f(p_n) \text{ and the ``écart''}$$

$$(p, p_n) = \frac{1}{(n+1)!}\frac{|x_{n+1}|}{1+|x_{n+1}|} + \cdots$$

approaches the limit 0 with $1/n$. Or, since $f(p)$ is continuous,

$$f(p) = \lim f(p_n) = \lim f_n(p).$$

3. **Measure of a Set.** Corresponding to any set of points, E, contained in the interval E_0, $(0 \leq x_n \leq 1)(n = 1, 2, \cdots)$, we can define a function equal to 1 on E, and equal to 0 otherwise. We define the measure of a set as the integral of this corresponding function (if it is summable). This measure will be non-negative, additive, and bounded. It is convenient to introduce a symbol "l" to denote either of the inequalities $<$, $\leqq$, and in what follows, it does not necessarily denote the same throughout. The function corresponding to the interval, $a_1 l x_1 l b_1, \cdots, a_n l x_n l b_n$, $0 \leq x_{n+1} \leq 1$, $0 \leq x_{n+2} \leq 1, \cdots$ (where $0 \leq a_1 \leq b_1 \leq 1, \cdots 0 \leq a_n \leq b_n \leq 1$), is a limit of functions, bounded in their set, belonging to class T_0, and the measure of this interval is

$$(b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

If we are given a sequence of such intervals, the measure of the limiting interval

$$a_1 l x_1 l b_1, \cdots a_n l x_n l b_n, a_{n+1} l x_{n+1} l b_{n+1}, \cdots,$$

is the limit of the infinite product,

$$(b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n) \cdots.$$

Let $u_n = 1 - b_n + a_n$, then $0 \leq u_n \leq 1$ and the product becomes

$$(1 - u_1)(1 - u_2) \cdots (1 - u_n) \cdots.$$

This infinite product converges if the series of non-negative terms $u_1 + u_2 + \cdots + u_n + \cdots$ converges, and diverges to 0, if the series is divergent. In either case the interval is measurable.

For example, the interval

$$A \ (0 \leqq x_n < 1)(n = 1, 2, \cdots),$$

is measurable and has the measure 1, while the interval

$$E_\epsilon \ (0 \leqq x_n \leqq 1 - \epsilon)(n = 1, 2, \cdots)(\epsilon > 0),$$

has the measure 0 however small the positive ϵ may be. It would seem

as if the intervals E_ϵ approach A as ϵ approaches 0, leading to a contradiction, but this is not the case. Consider the point

$$p = (\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \cdots).$$

This point lies in A, but no $\epsilon > 0$ can be found such that p lies in E_ϵ. In fact, A is rather the outer limiting set of all sets of the type

$$(0 \leqq x_n \leqq 1 - \epsilon_n)(n = 1, 2, \cdots),$$

and we can make the measure of this interval approach 1 as nearly as we please, by making the series $\Sigma\epsilon_n$ convergent and its sum sufficiently small.

Even in a triple integral, it is difficult to find examples, in which the integration can be performed analytically, if we disregard cases which reduce to simple integrals immediately. It is yet more difficult for an infinitely multiple integral. The following example is unsatisfactory because it is obtained by an infinite series of simple integrals. Let A be a point contained in E_0,

$$A = (a_1, a_2, \cdots, a_n, \cdots),$$

then consider

$$f(p) = \text{écart}\ (p, A) = \frac{|x_1 - a_1|}{1 + |x_1 - a_1|} + \cdots + \frac{1}{n!}\frac{|x_n - a_n|}{1 + |x_n - a_n|} + \cdots.$$

$I(f)$ will be

$$e - 1 - \sum_{n=1}^{\infty} \frac{1}{n!} \log(1 + a_n)(2 - a_n).$$

If $A = (0, 0, \cdots)$ or $(1, 1, 1, \cdots)$,

$$I(f) = (e - 1)(1 - \log 2).$$

4. **Multiple Stieltjes integral of positive type.** Let $\beta_1(t), \beta_2(t), \cdots$ be any denumerable set of functions of t, defined and nondecreasing from $t = -\infty$ to $t = +\infty$, and such that

$$\beta_n(-\infty) = \lim_{t \doteq -\infty} \beta_n(t) = 0,$$

$$\beta_n(+\infty) = \lim_{t \doteq +\infty} \beta_n(t) = 1.$$

Lemma 1. Let $B_n(t)(t > 0)$ denote

$$\int_{-t}^{+t} d\beta_n(t) = \beta_n(t) - \beta_n(-t) \geqq 0.$$

Given any $\epsilon_n > 0$, we can find M_n so that

$$B_n(M_n) > 1 - \epsilon_n.$$

For $B_n(t) = 1 - [1 - \beta_n(t)] - \beta_n(-t)$, and

$$\lim_{t \doteq \infty} [1 - \beta_n(t)] = 0, \qquad \lim_{t \doteq \infty} \beta_n(-t) = 0.$$

Lemma 2. Denote $B_n(M_n)$ by B_n, then $0 \leq B_n \leq 1$, and if $M_i \leqq M_i'$,

$$M_j \leqq M_j', \cdots, M_r \leqq M_r'.$$

$$B_i'B_j' \cdots B_r' - B_iB_j \cdots B_r \leqq (B_i' - B_i) + (B_j' - B_j) + \cdots + (B_r' - B_r).$$

For

$$\begin{aligned} B_i'B_j' \cdots B_r' - B_iB_j' \cdots B_r' &= (B_i' - B_i)B_j' \cdots B_r' \\ &\leqq B_i' - B_i, \\ B_iB_j' \cdots B_r' - B_iB_jB_h' \cdots B_r' &= (B_j' - B_j)B_iB_k' \cdots B_r' \\ &\leqq B_j' - B_j, \end{aligned}$$

and so on.

5. **Definition of integral.** As before we choose the class T_0 to be the class of functions of a finite number of variables $(x_i, \cdots, x_r)$, bounded and continuous for all finite values of these variables. We define

$$I(f) = \int_{-\infty}^{+\infty} \cdots \int f(x_i, x_j, \cdots, x_r) d\beta(x_i) \cdots d\beta(x_r).$$

This definition is possible since f is continuous and bounded, and

$$\beta(x_i, x_j, \cdots, x_r) = \beta(x_i)\beta(x_j) \cdots \beta(x_r)$$

is a limited function of positive type, i. e., such that

$$\Delta_i\Delta_j \cdots \Delta_r\beta(x_i, \cdots, x_r) \geq 0.$$

To justify the infinite limits we denote

$$\int_{-M_i}^{+M_i} \int_{-M_j}^{+M_j} \cdots \int_{-M_r}^{+M_r} f d\beta(x_i) \cdots d\beta(x_r) = I(f; M_i, \cdots M_r).$$

Then, if $M_i' \geqq M_i, \cdots M_r' \geqq M_r$,

$$\begin{aligned} | I(f; M_i', M_j', \cdots M_r') - I(f; M_i, M_j, \cdots M_r) | \\ &\leqq \max |f| \cdot [B_i'B_j' \cdots B_r' - B_iB_j \cdots B_r] \\ &\leqq \max |f| \cdot [(B_i' - B_i) + \cdots + (B_r' - B_r)], \end{aligned}$$

by Lemma 2. Then by Lemma 1, given any $\epsilon > 0$, we can find $M_i, M_j, \cdots, M_r$ so that for all $M_i' \geqq M_i, \cdots, M_r' \leqq M_r$, the last expression in a bracket is less than ϵ. The integral, so defined, satisfies the conditions $(C)(A)(P)$ and

$$| I(f) | \leqq \max |f(p)|.$$

To prove that condition (L) is also satisfied, we know, in the first place, that given any set of positive numbers $M_1, M_2, \cdots M_n, \cdots$, the domain

$$| x_n | \leq M_n, \qquad (n = 1, 2, \cdots)$$

is a finite domain (F., p. 42) and is therefore compact. By the same method of reasoning as we employed before, we can prove that given any $\epsilon > 0$ and any set of positive numbers $M_1, M_2, \cdots, M_n, \cdots$, we can find q_0, so that

$$I(f_q;\ M_1, \cdots) < \tfrac{1}{2}\epsilon \qquad (q \geqq q_0).$$

Choose the numbers $M_1, M_2, \cdots$, using lemma 1, so that,

$$B_n(M_n) > 1 - 3^{-n}\eta, \qquad (n = 1, 2, \cdots).$$

Then if f_q is a function of class T_0, and $f_q \geqq 0$,

$$\begin{aligned} I(f_q) - I(f_q;\ M_1, M_2, \cdots) &< \max_p f_q(p)\cdot\eta(3^{-1} + 3^{-2} + \cdots) \\ &< \tfrac{1}{2}\eta \max_p f_q(p), \\ &< \tfrac{1}{2}\eta \max_p f_1(p)\cdot(q = 1, 2, 3, \cdots). \end{aligned}$$

Choose $\eta = \epsilon \div \max f_1(p)$, then we can choose $M_1, M_2, \cdots$, so that *for all q*

$$I(f_q) - I(f_q;\ M_1, M_2, \cdots) < \tfrac{1}{2}\epsilon,$$

and then with this choice made we can find q_0 so that

$$I(f_q;\ M_1, M_2, \cdots) < \tfrac{1}{2}\epsilon \qquad (q \geqq q_0).$$

Then combining these, given any $\epsilon > 0$ we can find q_0, so that

$$I(f_q) < \epsilon \quad (q \geqq q_0),$$

or limit $I(f_n) = 0$.

As before, we can extend the definition to all continuous, bounded functions and to all functions obtained by a succession of operations of addition, multiplication by a constant, taking the modulus, and taking the limit of a sequence of functions bounded in their set.

Example. Let

$$\beta_n(t) = \int_{-\infty}^{t} e^{-\pi t^2} dt, \qquad (n = 1, 2, \cdots);$$

then

$$\beta_n(\infty) = \int_{-\infty}^{+\infty} e^{-\pi t^2} dt = 1.$$

If $f(p)$ is any bounded continuous function in E_ω, the integral

$$I(f) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \cdots f(x_{,1}\, x_2, \cdots x_n, \cdots) e^{-\pi \sum_1^\infty x_n^2} dx_1 \cdots dx_n \cdots$$

can be defined, and we may add the convention that when Σx_n^2 is divergent,

$$e^{-\pi \sum_1^\infty x_n^2} = 0.$$

If

$$\begin{aligned} \beta_n(t) &= 0, & t &< 0, \\ &= t, & 0 \leq t &\leq 1, \qquad (n = 1, 2, \cdots) \\ &= 1, & t &> 1, \end{aligned}$$

then this more general integral reduces to the case first considered.

Note. In finding an example of the simple integral in an infinite number of dimensions, the author desired to invent one which could readily be evaluated. For this reason it was somewhat unsatisfactory. But if we wish a genuine example and do not require its evaluation we may set up the integral,

$$I(f) = \int_0^1 \cdots \int_0^1 \cdots f(p) dx_1 dx_2 \cdots dx_n \cdots,$$

where

$$f(p) = [e - (p, A)]^{1/2}$$

(p, A) = écart between p and a fixed point A.

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